

# A NOTE ON THE HYBRID STEEPEST DESCENT METHODS

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**ABSTRACT.** The aim of this paper is to prove that, in an appropriate setting, every iterative sequence generated by the hybrid steepest descent method is convergent whenever so is every iterative sequence generated by the Halpern type iterative method.

## 1. INTRODUCTION

In this paper, we consider the following variational inequality problem in a real Hilbert space  $H$ : Find  $z \in F$  such that

$$\langle y - z, Az \rangle \geq 0 \text{ for all } y \in F,$$

where  $F$  is the set of common fixed points of a sequence  $\{T_n\}$  of nonexpansive mappings on  $H$  and  $A$  is a strongly monotone and lipschitzian mapping on  $H$ . Then we study convergence of the iterative sequence  $\{x_n\}$  defined by  $x_1 \in H$  and

$$(1.1) \quad x_{n+1} = (I - \lambda_n A)T_n x_n$$

for  $n \in \mathbb{N}$  in order to approximate the solution, where  $I$  is the identity mapping on  $H$ . If  $A = I - u$  for some  $u \in H$ , then it is clear that  $A$  is strongly monotone and lipschitzian, and (1.1) is reduced to

$$(1.2) \quad x_{n+1} = \lambda_n u + (1 - \lambda_n)T_n x_n.$$

We deal with these two types of iterations, and especially we focus on the relationship between them; see §3.

The iterative method defined by (1.1) is called the hybrid steepest descent method, which was introduced by Yamada [32]. He considered the variational inequality problem over the set of common fixed points of a finite family of nonexpansive mappings and proved strong convergence of the sequence generated by the method. We know many results by using the hybrid steepest descent method; see [2, 10–12, 14, 15, 17–20, 25, 30, 31, 33–38].

The iterative method defined by (1.2) is called the Halpern type iterative method; see Halpern [13], Wittmann [27], and Shioji and Takahashi [21]; see also [1, 4, 5].

## 2. PRELIMINARIES

Throughout the present paper,  $H$  denotes a real Hilbert space with the inner product  $\langle \cdot, \cdot \rangle$  and the norm  $\|\cdot\|$ ,  $I$  the identity mapping on  $H$ , and  $\mathbb{N}$  the set of positive integers.

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Let  $C$  be a nonempty closed convex subset of  $H$ . A mapping  $S: C \rightarrow C$  is said to be lipschitzian if there exists a constant  $\eta > 0$  such that  $\|Sx - Sy\| \leq \eta \|x - y\|$  for all  $x, y \in C$ . In this case,  $S$  is called an  $\eta$ -lipschitzian mapping. In particular, an  $\eta$ -lipschitzian mapping is said to be nonexpansive if  $\eta = 1$ ; an  $\eta$ -lipschitzian mapping is said to be a contraction if  $0 \leq \eta < 1$ . It is known that  $\text{Fix}(S)$  is closed and convex if  $S$  is nonexpansive, where  $\text{Fix}(S)$  denotes the set of fixed points of  $S$ . The metric projection of  $H$  onto  $C$  is denoted by  $P_C$  and we know that  $P_C$  is nonexpansive. We also know the following; see [23].

**Lemma 2.1.** *Let  $x \in H$  and  $z \in C$ . Then  $z = P_C(x)$  if and only if  $\langle y - z, x - z \rangle \leq 0$  for all  $y \in C$ .*

Let  $C$  be a nonempty closed convex subset of  $H$ . Let  $\{S_n\}$  be a sequence of self-mappings of  $C$ . We say that  $\{S_n\}$  satisfies the condition (Z) if every weak cluster point of  $\{x_n\}$  is a common fixed point of  $\{S_n\}$  whenever  $\{x_n\}$  is a bounded sequence in  $C$  and  $x_n - S_n x_n \rightarrow 0$ ; see [1, 3, 6–9].

A mapping  $A: H \rightarrow H$  is said to be strongly monotone if there is a constant  $\kappa > 0$  such that  $\langle x - y, Ax - Ay \rangle \geq \kappa \|x - y\|^2$  for all  $x, y \in H$ . In this case,  $A$  is called a  $\kappa$ -strongly monotone mapping. In §3, we deal with the following variational inequality problem:

**Problem 2.2.** *Let  $\kappa$  and  $\eta$  be positive real numbers such that  $\eta^2 < 2\kappa$ . Let  $F$  a nonempty closed convex subset of  $H$  and  $A: H \rightarrow H$  a  $\kappa$ -strongly monotone and  $\eta$ -lipschitzian mapping. Then find  $z \in F$  such that*

$$\langle y - z, Az \rangle \geq 0 \text{ for all } y \in F.$$

The set of solution of Problem 2.2 is denoted by  $\text{VI}(F, A)$ . It is known that the solution set is a singleton; see Lemma 2.4 below.

*Remark 2.3.* The assumption that  $\eta^2 < 2\kappa$  in Problem 2.2 is not restrictive. Indeed, suppose that a  $\kappa$ -strongly monotone and  $\eta$ -lipschitzian mapping  $A$  is given. Let us choose a positive constant  $\mu$  such that  $\mu < 2\kappa/\eta^2$ , and define  $\kappa' = \mu\kappa$  and  $\eta' = \mu\eta$ . Then it is easy to verify that  $(\eta')^2 < 2\kappa'$ ,  $\mu A$  is  $\kappa'$ -strongly monotone and  $\eta'$ -lipschitzian, and moreover,  $\text{VI}(F, A) = \text{VI}(F, \mu A)$  for every nonempty closed convex subset  $F$  of  $H$ .

**Lemma 2.4.** *Under the assumptions of Problem 2.2, the following hold:*

- (1)  $\kappa \leq \eta$ ,  $0 \leq 1 - 2\kappa + \eta^2 < 1$  and  $I - A$  is a  $\theta$ -contraction, where  $\theta = \sqrt{1 - 2\kappa + \eta^2}$ .
- (2) Problem 2.2 has a unique solution and  $\text{VI}(F, A) = \text{Fix}(P_F(I - A))$ .

*Proof.* We first prove (1). Since  $A$  is  $\kappa$ -strongly monotone and  $\eta$ -lipschitzian, it follows that

$$\kappa \|x - y\|^2 \leq \langle x - y, Ax - Ay \rangle \leq \|x - y\| \|Ax - Ay\| \leq \eta \|x - y\|^2$$

and

$$\begin{aligned} \|(I - A)x - (I - A)y\|^2 &= \|x - y\|^2 - 2\langle x - y, Ax - Ay \rangle + \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\kappa \|x - y\|^2 + \eta^2 \|x - y\|^2 \\ &= (1 - 2\kappa + \eta^2) \|x - y\|^2 \end{aligned}$$

for all  $x, y \in H$ . Therefore,  $\kappa \leq \eta$ . By assumption, it is easy to check that  $0 \leq 1 - 2\kappa + \eta^2 < 1$ , and thus  $I - A$  is a  $\theta$ -contraction.

We next prove (2). Since  $I - A$  is a contraction by (1) and the metric projection  $P_F$  is nonexpansive,  $P_F(I - A)$  is a contraction on  $H$ . The Banach contraction principle guarantees that  $P_F(I - A)$  has a unique fixed point. On the other hand, Lemma 2.1 shows that  $\text{VI}(F, A) = \text{Fix}(P_F(I - A))$  and thus Problem 2.2 has a unique solution.  $\square$

We know the following result; see [2, 5] and see also [3, 9].

**Theorem 2.5.** *Let  $H$  be a Hilbert space,  $C$  a nonempty closed convex subset of  $H$ ,  $\{T_n\}$  a sequence of nonexpansive self-mappings of  $C$  with a common fixed point,  $F$  the set of common fixed points of  $\{T_n\}$ , and  $\{\lambda_n\}$  a sequence in  $[0, 1]$  such that*

$$(2.1) \quad \lambda_n \rightarrow 0, \sum_{n=1}^{\infty} \lambda_n = \infty, \text{ and } \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

*Suppose that  $\{T_n\}$  satisfies the condition (Z) and*

$$(2.2) \quad \sum_{n=1}^{\infty} \sup\{\|T_{n+1}y - T_ny\| : y \in D\} < \infty$$

*for every nonempty bounded subset  $D$  of  $C$ . Let  $x, u$  be points in  $C$  and  $\{x_n\}$  a sequence defined by  $x_1 = x$  and (1.2) for  $n \in \mathbb{N}$ . Then  $\{x_n\}$  converges strongly to  $P_F(u)$ .*

We also know the following result; see [1, 4].

**Theorem 2.6.** *Let  $H$  be a Hilbert space,  $C$  a nonempty closed convex subset of  $H$ ,  $\{S_n\}$  a sequence of nonexpansive self-mappings of  $C$  with a common fixed point,  $F$  the set of common fixed points of  $\{S_n\}$ . Let  $\{\lambda_n\}$  and  $\{\beta_n\}$  be a sequences in  $[0, 1]$  such that*

$$\lambda_n \rightarrow 0, \sum_{n=1}^{\infty} \lambda_n = \infty, \text{ and } 0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1.$$

*Suppose that  $\{S_n\}$  satisfies the condition (Z) and*

$$(2.3) \quad \sup\{\|S_{n+1}y - S_ny\| : y \in D\} \rightarrow 0$$

*for every nonempty bounded subset  $D$  of  $C$ . Let  $x, u$  be points in  $C$  and  $\{x_n\}$  a sequence defined by  $x_1 = x$  and*

$$x_{n+1} = \lambda_n u + (1 - \lambda_n)(\beta_n x_n + (1 - \beta_n)S_n x_n)$$

*for  $n \in \mathbb{N}$ . Then  $\{x_n\}$  converges strongly to  $P_F(u)$ .*

The following lemma is well known; see [5, 16, 26, 28, 29].

**Lemma 2.7.** *Let  $\{\epsilon_n\}$  be a sequence of nonnegative real numbers,  $\{\gamma_n\}$  a sequence of real numbers, and  $\{\lambda_n\}$  a sequence in  $[0, 1]$ . Suppose that  $\epsilon_{n+1} \leq (1 - \lambda_n)\epsilon_n + \lambda_n\gamma_n$  for every  $n \in \mathbb{N}$ ,  $\limsup_{n \rightarrow \infty} \gamma_n \leq 0$ , and  $\sum_{n=1}^{\infty} \lambda_n = \infty$ . Then  $\epsilon_n \rightarrow 0$ .*

### 3. CONVERGENCE THEOREMS BY THE HYBRID STEEPEST DESCENT METHOD

In this section, we deal with the variational inequality problem over the set of common fixed points of a sequence of nonexpansive mappings; see Problem 3.1 below. We first investigate the relationship between the hybrid steepest descent method and the Halpern type iterative method (Theorem 3.2). Then, by using Theorems 2.5 and 2.6, we show two convergence theorems by the hybrid steepest descent method for this problem.

**Problem 3.1.** *Let  $H$  be a Hilbert space,  $\{T_n\}$  a sequence of nonexpansive self-mappings of  $H$  with a common fixed point,  $F$  the set of common fixed points of  $\{T_n\}$ , and  $A: H \rightarrow H$  a  $\kappa$ -strongly monotone and  $\eta$ -lipschitzian mapping, where  $\kappa$  and  $\eta$  are positive real numbers such that  $\eta^2 < 2\kappa$ . Then find  $z \in F$  such that*

$$\langle y - z, Az \rangle \geq 0 \text{ for all } y \in F.$$

Using the technique in [22], we can prove the following theorem, which shows that every sequence generated by the hybrid steepest descent method for Problem 3.1 is convergent whenever so is every sequence generated by the Halpern type iterative method for the sequence of nonexpansive mappings.

**Theorem 3.2.** *Let  $H$ ,  $\{T_n\}$ ,  $F$ ,  $\kappa$ ,  $\eta$ , and  $A$  be the same as in Problem 3.1. Let  $\{\lambda_n\}$  be a sequence in  $[0, 1]$  such that  $\sum_{n=1}^{\infty} \lambda_n = \infty$ . Suppose that for any  $(x, u) \in H \times H$ , the sequence  $\{x_n\}$  defined by  $x_1 = x$  and*

$$(3.1) \quad x_{n+1} = \lambda_n u + (1 - \lambda_n) T_n x_n$$

*for  $n \in \mathbb{N}$  converges strongly to  $P_F(u)$ . Let  $y$  be a point in  $H$  and  $\{y_n\}$  a sequence defined by  $y_1 = y$  and*

$$(3.2) \quad y_{n+1} = (I - \lambda_n A) T_n y_n$$

*for  $n \in \mathbb{N}$ . Then  $\{y_n\}$  converges strongly to the unique solution of Problem 3.1.*

*Proof.* Set  $f_n = (I - A)T_n$  for  $n \in \mathbb{N}$ . Since  $T_n$  is nonexpansive,  $f_n$  is a  $\theta$ -contraction on  $H$  by Lemma 2.4, where  $\theta = \sqrt{1 - 2\kappa + \eta^2}$ . Let  $w$  be the fixed point of  $P_F \circ f_1$  and  $\{x_n\}$  a sequence defined by  $x_1 = y$  and

$$x_{n+1} = \lambda_n f_1(w) + (1 - \lambda_n) T_n x_n$$

for  $n \in \mathbb{N}$ . Then

$$(3.3) \quad x_n \rightarrow P_F(f_1(w)) = w$$

by assumption. Since  $T_n$  is nonexpansive and  $f_n$  is a  $\theta$ -contraction, it follows from  $f_1(w) = f_n(w)$  that

$$\begin{aligned} \|x_{n+1} - y_{n+1}\| &= \|(1 - \lambda_n)(T_n x_n - T_n y_n) + \lambda_n(f_1(w) - f_n(y_n))\| \\ &\leq (1 - \lambda_n) \|T_n x_n - T_n y_n\| + \lambda_n \|f_n(w) - f_n(y_n)\| \\ &\leq (1 - \lambda_n) \|x_n - y_n\| + \lambda_n \theta \|w - y_n\| \\ &\leq (1 - \lambda_n) \|x_n - y_n\| + \lambda_n \theta (\|w - x_n\| + \|x_n - y_n\|) \\ &\leq (1 - (1 - \theta)\lambda_n) \|x_n - y_n\| + (1 - \theta)\lambda_n \frac{\theta}{1 - \theta} \|x_n - w\| \end{aligned}$$

for every  $n \in \mathbb{N}$ . Since  $\sum_{n=1}^{\infty} (1 - \theta)\lambda_n = \infty$ , (3.3) and Lemma 2.7 show that  $x_n - y_n \rightarrow 0$ . Therefore, we conclude that  $\{y_n\}$  converges strongly to  $w = P_F((I -$

$A)T_1w) = P_F(I - A)w$ , which is the unique solution of Problem 3.1 by Lemma 2.4. This completes the proof.  $\square$

Using Theorems 2.5 and 3.2, we obtain the following:

**Theorem 3.3.** *Let  $H$ ,  $\{T_n\}$ ,  $F$ ,  $\kappa$ ,  $\eta$ , and  $A$  be the same as in Problem 3.1. Let  $\{\lambda_n\}$  be a sequence in  $[0, 1]$  such that (2.1) holds. Suppose that  $\{T_n\}$  satisfies the condition (Z) and (2.2) holds for every nonempty bounded subset  $D$  of  $C$ . Let  $y$  be a point in  $H$  and  $\{y_n\}$  a sequence defined by  $y_1 = y$  and (3.2) for  $n \in \mathbb{N}$ . Then  $\{y_n\}$  converges strongly to the unique solution of Problem 3.1.*

*Proof.* Let  $(x, u) \in H \times H$  be fixed. Then it follows from Theorem 2.5 that the sequence  $\{x_n\}$  defined by  $x_1 = x$  and (3.1) for  $n \in \mathbb{N}$  converges strongly to  $P_F(u)$ . Therefore, Theorem 3.2 implies the conclusion.  $\square$

Using Theorem 3.2 and other known results, we also obtain the following:

**Theorem 3.4** (Iemoto and Takahashi [14, Theorem 3.1]). *Let  $H$ ,  $\{T_n\}$ ,  $F$ ,  $\kappa$ ,  $\eta$ , and  $A$  be the same as in Problem 3.1. Let  $\{\lambda_n\}$  be a sequence in  $[0, 1]$  such that*

$$\lambda_n \rightarrow 0 \text{ and } \sum_{n=1}^{\infty} \lambda_n = \infty$$

*and  $\{\gamma_n\}$  a sequence in  $[a, b]$ , where  $0 < a \leq b < 1$ . For each  $n \in \mathbb{N}$  and  $k \in \{1, 2, \dots, n+1\}$ , define a mapping  $U_{n,k}$  by  $U_{n,n+1} = I$  and*

$$U_{n,k} = \gamma_k T_k U_{n,k+1} + (1 - \gamma_k)I.$$

*Let  $y$  be a point in  $H$  and  $\{y_n\}$  a sequence defined by  $y_1 = y$  and*

$$(3.4) \quad y_{n+1} = (I - \lambda_n A)U_{n,1}y_n$$

*for  $n \in \mathbb{N}$ . Then  $\{y_n\}$  converges strongly to the unique solution of Problem 3.1.*

*Proof.* Set  $S_n = T_1 U_{n,2}$  for  $n \in \mathbb{N}$ . Then it is clear that each  $S_n$  is nonexpansive. It is known that

$$\text{Fix}(S_n) = \text{Fix}(U_{n,1}) = \bigcap_{k=1}^n \text{Fix}(T_k)$$

by [24, Lemma 3.2]; see also [7, Lemma 4.2]. Hence we have

$$\bigcap_{n=1}^{\infty} \text{Fix}(S_n) = \bigcap_{n=1}^{\infty} \text{Fix}(U_{n,1}) = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^n \text{Fix}(T_k) = F.$$

It is also known that  $\{S_n\}$  satisfies the condition (Z) and (2.3) holds for every nonempty bounded subset  $D$  of  $H$ ; see [7], [9], [3], and [4]. Thus, for any  $(x, u) \in H \times H$ , it follows from Theorem 2.6 that the sequence  $\{x_n\}$  defined by  $x_1 = x$  and

$$x_{n+1} = \lambda_n u + (1 - \lambda_n)U_{n,1}x_n = \lambda_n u + (1 - \lambda_n)((1 - \gamma_1)x_n + \gamma_1 S_n x_n)$$

for  $n \in \mathbb{N}$  converges strongly to  $P_F(u)$ . Therefore, Theorem 3.2 implies the conclusion.  $\square$

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